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The flow in and around air-bubble plumes

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Abstract

(1) A short review is given of some of the important publications on air-bubble plumes. Two main ways of modelling, using either the entrainment assumption, or the energy balance principle, are presented and briefly compared. (2) Focusing attention on the entrainment coefficient $\alpha(z)$, we show how useful information about this coefficient can be found, for a plane plume, from the characteristics of the external flow (the "return flow"). (3) A theory is presented for the external flow far outside a plane plume, at horizontal distances $x \geqslant 3D$, D being the water depth. (4) A similar theory is presented for the surface flow close to the plume (both axisymmetric and plane cases covered). Our theory replaces the physically non-permissible assumption about conservation of momentum around the turning region, used in earlier approaches, with an energy condition in which turbulent dissipation is included. Comparisons of the theories with experiments reported in the literature show reasonable agreement.

Keywords: Two-phase flow; Air-bubble plumes; Turbulence

1. Introduction and summary

The injection of air bubbles into water in order to produce a current is a method that has been in use for at least a century. Let us summarize some engineering applications of such a system:

1. Production of surface currents to protect harbour areas against high amplitude waves (Taylor, 1955; Bulson, 1961, 1963, 1968; Brevik, 1976a,b).
2. Air injection in drinking water reservoirs to prevent growth of algae (Schladow, 1992).

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3. Prevention of oil slicks from spreading after oil tanker accident, or protection of coastal habitats against damage from oil (Fanneløp, 1994, Ch. 7).
4. Production of circulation in fjords to make dirty water rise to the surface where pollutants can evaporate. This technique is efficiently made use of, in Norwegian fjords.
5. Production of air-lift in an oil reservoir to improve the efficiency.
6. Moving large floating objects, such as icebergs (Riess and Fanneløp, 1995).
7. Finally, as a rather esoteric possibility, the use of air plumes to explore mineral deposits beneath the sea floor.

Let us briefly consider some of the earlier works. As for engineering models of air-bubble plumes, a significant early proposal was made by Taylor (1955). He exploited the analogies with turbulent jet theory and with heat plume theory. His theory, as well as the later ones of engineering interest, has been phenomenological. Computational fluid dynamical methods (CFD) in the context of air-bubble plumes – cf., for instance, Bernard et al. (2000) – are evidently of fundamental interest, but they have poor accuracy, especially outside the plume, at present.

For the case of *plane* plumes, the large scale experiments of Bulson (1961, 1963, 1968) are important. He measured the horizontal currents produced by a linear source at a depth up to $D = 10$ m, and found empirical formulas for the velocity, and the thickness, of this horizontal branch. Kobus (1968) developed the first fairly detailed analytical model of the bubble plume, and he carried out a series of experiments, at $D = \{2, 4.3, 10.4\}$ m. He took the vertical velocity profile to be Gaussian, with a width linearly increasing with height $(z + z_0)$ above the virtual source. Here z_0 is the depth of the virtual source beneath the real source. Weak points in Kobus' approach are the following: first, his experiments were made with very small orifices, of diameter $d = 0.1$ cm, so that supercritical flow could result (cf. also the remarks of Milgram, 1983). Second, his way of calculating the buoyant force is questionable. Another phenomenological theory was given by Ditmars and Cederwall (1974). They started from single-phase buoyant plume theory, and included isothermal air-bubble expansion as well as a bubble slip velocity w_{rel}. The same profiles for vertical fluid velocity and density deficiency were assumed as in Kobus' paper. Moreover, they assumed the rate of entrainment to be proportional to the mean centerline velocity:

$$\frac{\mathrm{d}Q_{\mathrm{w}}}{\mathrm{d}z} = 2\alpha w_{\mathrm{m}}. \tag{1}$$

Here Q_{w} is the water volume flux. The entrainment parameter α was assumed to be constant (they found $\alpha \approx 0.1$). Eq. (1), together with the momentum conservation equation, formed a set of two integral equations which were integrated numerically. A different approach to the plane air-bubble case was that of Brevik (1977), the main difference in comparison with the theory of Cederwall and Ditmars (1970) being the use of the kinetic energy equation instead of the entrainment equation (1). The kinetic energy equation assumed self-preservation of turbulent velocity fluctuations:

$$\frac{\overline{w'u'}}{w_{\mathrm{m}}^2(z)} = f(\eta) \tag{2}$$

in standard notation, where $f(\eta)$ is an unspecified function of the parameter $\eta = x/\sigma(z)$, $\sigma(z)$ being the standard deviation. Fanneløp et al. (1991), in an extensive article, focused on the surface current and recirculating cells outside the plume, and gave also a theory of the line-source bubble plume itself. The recent article of Riess and Fanneløp (1998) – cf. also the thesis of Riess (1997) – follows the same kind of approach and presents a return flow theory based on a jet going out from the plume and a uniform return flow. The paper of Brevik and Kluge (1999) focused on the turbulent kinetic energy flow, a factor usually being neglected in prior works. A parameter k, defined as

$$k = \frac{\overline{w'^2(z,x)}}{\overline{w}^2(z,x)} = \text{constant}, \tag{3}$$

gives the ratio between vertical turbulent energy and vertical mean kinetic energy. On experimental grounds, Goossens (1979) estimated $k \approx 0.3$. Related papers are those of Brevik (1977) and Brevik and Killie (1996).

Proceeding to the case of *axisymmetric* plumes, the thesis of Goossens (1979) – containing both theoretical and experimental contributions – plays an important role. Omitting his minor "hold up" effect, as well as the effect of turbulent fluctuations, and transforming back to Gaussian profiles, one recovers the same momentum and entrainment equations as Ditmars and Cederwall (1974). The key parameter is the entrainment parameter α, being about 0.056 under usual conditions. The report of Fanneløp and Sjøen (1980), describing a Norwegian underwater blowout project, is the basis of the later theory of Fanneløp et al. (1991). This study was concerned with deepset plumes in the North Sea, in particular Topham's experiments at $D = 53$ m and $D = 60$ m (cf. Topham, 1975). Another extensive contribution is the paper of Milgram (1983), reporting on full scale experiments with $D = 50$ m. Finally, we mention that a useful overview of the closure conditions employed in the phenomenological theories is given in the report of Haaland (1979).

The crown, or boil region, was in focus in Engebretsen et al. (1997) and the thesis of Friedl (1998). This is especially interesting in the case of violent underwater blowouts near oil rigs. The three-dimensional oscillation of the plume in a vertical cylinder was investigated by Kuwagi and Ozoe (1999) carrying out both experiments and numerical simulations. A more fundamental investigation into bubbles rising in line was performed by Ruzicka (2000) showing that a nearest-neighbour approximation was satisfactory except for the first bubbles in the line.

Chemical bubble column reactors have recently been subject to many studies where simulations with CFD have been compared with experiments. The approach is microscopical rather than macroscopical, and this field is rather detached form the other studies of air-bubble plumes. The thesis of Grevskott (1997) and the articles in issue 21, volume 54 of Chemical Engineering Science are within this field. Mudde and Saito (2001) looked at the similarities between a bubble column and bubbly pipe flow, finding that in many respects the bubbly pipe flow is the superposition of the flow in the bubble column and the single-phase flow which would exist without the bubbles.

Our motivation for undertaking the present work is the following. First, we will replace the entrainment assumption with an energy balance principle. The entrainment assumption is that the mean flow across the edge of the plume is proportional to some characteristic velocity inside the plume, usually taken to be the mean centerline velocity. For thermal plumes this is sound, in the sense that it can be found that for self-similar plumes the assumption is consistent with the conservation equations. For air-bubble plumes the assumption appears not to agree so well with what is found experimentally. The reason for this is not clear. It may seem natural to investigate, therefore, alternatives to the entrainment assumption. Perhaps can a consideration of the kinetic energy balance be a better approach. The energy approach was developed by Brevik (1977). This method has become more plausible in the light of new experimental evidence; as we will see in Section 2.2, the entrainment approach gives results lying even farther from the experiments.

Next, another main motivation for the present study is to construct a model for the flow *outside* a plane plume. We assume a Gaussian velocity profile above, as well as below, the stagnation line, and introduce a dissipation factor in order to deal with turbulent dissipation in the turning region. With parameter values inferred from experimental data, we obtain in this way a theory that ought to be of engineering interest.

We briefly summarize our results:

1. A relation is given between the entrainment parameter $\alpha(z)$ and the return flow for the plane plume case; see Eq. (6). A rough evaluation of this formula using the return flow model of Riess and Fanneløp (1998) and centerline velocities and widths from Kobus (1968) gives reasonable results. In this model the return flow is uniform, and $\alpha(z)$ is found to be almost constant.
2. A simple formula for estimating α for the plane plume is given by Eq. (12). This formula should be of immediate engineering interest.
3. A theory for the flow far outside a plane plume, including an equation for the position of the stagnation line, is given in Section 3. An advantage of this kind of approach is that no empirical input values for the spreading factor k, and momentum factor m^*, present in the return-field model of Riess and Fanneløp (1998), are now needed. A new element in our model is that the return flow is allowed to be *non-uniform*.
4. A theory for the surface flow close to the plume, both in the axisymmetric case and in the plane case, making use of the balance of kinetic energy and accounting for turbulent energy losses through a dissipation factor f, is given in Section 4. Calculated results are shown in Fig. 4 for the axisymmetric case, and in Fig. 6 for the plane case, both figures showing reasonable agreement with observations.

2. Remarks on the entrainment hypothesis

The simple entrainment hypothesis has been remarkably successful, and experimental results over a range in depth in excess of 400 can be well predicted by this model. The smallest experiments have a depth of only about 20 cm, whereas the deepest is Milgram's 50 m tests. The entrainment coefficient α is usually assumed to be independent of the height.

Although a drawback of the entrainment model is that we do not understand why it works so well, it seems reasonable to try to determine the value of α via different routes. An idea presented below is to make use of the return flow model to estimate α in the line-source case. This idea is related to the approach of Leitch and Baines (1989).

2.1. α Determined from the return flow

The definition equation for α is Eq. (1). The centerline velocity $w_m(z)$ should be precisely defined: it is the mean local liquid velocity. (The local gas velocity is accordingly the sum of w_m and the slip velocity w_{rel}, in accordance with Milgram, 1983.) In practice, to obtain mean values, it will usually be necessary to average over quite long time intervals, 3–5 min. In principle, the derivative dQ_w/dz of the water flux can be determined if the return flow outside the plume is known. This opens for a possibility of determining $\alpha(z)$.

Consider, in a plane plume, a segment of unit length along the source direction, of width $2b$ and height dz. It is natural to choose b equal to the horizontal distance at which the vertical velocity has decreased to $1/e$ of its centerline value. Invoking the equation of continuity for the water flow and integrating it over x from 0 to b for a given value of z we obtain, when combining with Eq. (1), the following expression:

$$\alpha(z) = -\frac{1.19u(z)|_{x=b}}{w_m(z)} \tag{4}$$

for the entrainment coefficient. We now invoke the return field model of Riess and Fanneløp (1998). According to this model one writes $u(z,x) = w_p u^*(z,x)$, where

$$u^*(z,x) = m^*\sqrt{\frac{D}{x}}\exp\left(-\frac{z^2}{k^2x^2}\right) - \frac{1}{2}km^*\sqrt{\frac{\pi x}{D}}\operatorname{erf}\left(\frac{D}{kx}\right), \tag{5}$$

$\operatorname{erf}(\cdot\cdot)$ being the error function. Here k and m^* are spreading and momentum factors found by least square fits to the measurements. Further, w_p is a scaling parameter equal to the centerline velocity of the plume at the surface. This procedure for finding w_p using Fanneløp's theory involves α, which will then have to be isolated. Following this procedure we find w_p to be proportional to $\alpha^{-1/3}$. Then defining the quantity s by $w_p = s\alpha^{-1/3}$, we arrive at the final formula

$$\alpha(z) = \left(-\frac{1.19su^*(z)|_{z=b}}{w_m(z)}\right)^{3/4}. \tag{6}$$

This formula for α can be evaluated: First, w_m is found from Kobus' formula for the centerline velocity:

$$w_m(z) = 1.75\left(gQ^0\right)^{1/3}\left[\frac{-P}{z+z_0}\ln\left(1-\frac{z}{D+P}\right)\right]^{1/2}. \tag{7}$$

Here P is the atmospheric pressure as a head of water, and Q^0 is the volume of air per meter of pipe and per second at atmospheric pressure. Next, b can be found from Kobus' formula for the spreading of the plume: $b = \sqrt{2}\cdot 0.18\cdot (Q^0)^{0.15}\cdot (z+z_0)$.

Alternatively, b can be determined by using the analytical model of Fanneløp et al. (1991). Ignoring slip velocity they introduced parameters Z and B defined as

$$Z = \frac{z}{D+P}, \quad B = \frac{\sqrt{\pi}b(z)}{2\alpha(D+P)}, \tag{8}$$

and also two new parameters W and M:

$$W = \frac{w_m(z)}{M}, \quad M = \left(\frac{gQ^0\sqrt{1+\lambda^2}}{\sqrt{2}\gamma\alpha}\right)^{1/3}. \tag{9}$$

Here λ has the usual meaning of being the ratio between the widths of density deficiency and vertical velocity. Further, $\gamma \approx 1.4$ is the turbulent momentum amplification factor. The authors gave their solution on polynomial form:

$$B(Z) = Z - \frac{1}{8}Z^2 - \frac{31}{480}Z^3 - \frac{2700}{25040}Z^4, \tag{10}$$

$$W(Z) = 1 + \frac{1}{4}Z + \frac{23}{160}Z^2 + \frac{1162}{5760}Z^3. \tag{11}$$

Fig. 1 shows plots of $\alpha(z)$, based on use of Eq. (6). The upper and lower parts refer to whether b is calculated from Kobus or Fanneløp et al. In the figure we assumed $Q^0 = 3.41 \times 10^{-3}$ m^2/s and $D = 1$ m, in order to use the model of Riess and Fanneløp (their values for k and m^* were given only for this case). These authors recommended $\alpha = 0.08$, so that this is the reference value shown by dotted lines. As for the value of z_0 we scaled Kobus' recommendation of $z_0 = 0.8$ m down somewhat, from $z_0 = 0.8$ m to $z_0 = 0.4$ m, since the present value of D is lower than that in Kobus' experiment (the choice of $z_0 = 0.6$ turned out to give similar results). The figure shows that α is in fact fairly constant, except in the vicinity of the sources, and lies close to the recommended value $\alpha = 0.08$. The fairly constant α is not surprising given that a uniform return flow model was used. The return flow model in Section 3.2 did not give acceptable values of α. The value of α is predicted to be higher when b is found from Kobus, since Kobus' b value is higher than Fanneløp's. The return flow field is then evaluated farther from the plume when the inward velocity is higher,

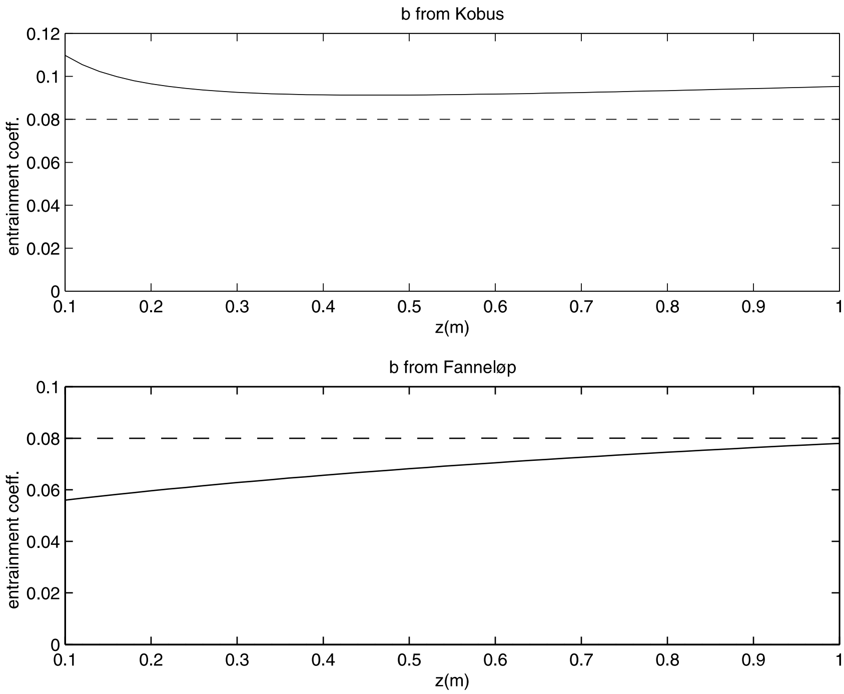

Fig. 1. Plot of $\alpha(z)$ using Eq. (6). $Q^0 = 3.41 \times 10^{-3}$ m^2/s, $D = 1$ m, $z_0 = 0.4$ m.

leading to a higher α. Using Fanneløp's analytical theory for both b and w_m yielded lower values of α.

2.2. Other ways of obtaining α – The energy approach

As mentioned, the kinetic energy approach by one of the present authors (Brevik, 1977) was not based on the entrainment hypothesis, but a formula for α was nevertheless derived in order to compare the theory with entrainment-based theories. Following the arguments of the mentioned paper the dependence on λ cancels out, leaving the expression $\alpha = \sqrt{\pi/2}\,\mathrm{d}\sigma/\mathrm{d}z$, σ being the standard deviation for the vertical velocity. Inserting Kobus' expression for the spreading coefficient, $c = \mathrm{d}\sigma/\mathrm{d}z = 0.18(Q^0)^{0.15}$, we get $\alpha = 0.226(Q^0)^{0.15}$, independent of z. Adopting from Haaland's report the formula $\alpha = 1.18\,\mathrm{d}\sigma/\mathrm{d}z$, we thus get

$$\alpha = 0.22(Q^0)^{0.15}. \tag{12}$$

We suggest that this simple formula is generally useful for engineering purposes. With the volume flow rate of Fig. 1 we get $\alpha = 0.096$, which is in reasonable agreement with the plots of Eq. (6) with b determined from Kobus. Of course, it is quite a rough approximation to put $\mathrm{d}\sigma/\mathrm{d}z = \mathrm{const}$. In practice, $\mathrm{d}\sigma/\mathrm{d}z$ is largest close to the bottom, causing the value of α to be largest in this zone. The effect is reproduced in Fig. 1(a).

Finally, let us make some remarks on the connection with the kinetic energy approach. In the latter theory, the slip velocity is taken into account. Most likely, the choice $w_{\mathrm{rel}} = 0.30$ m/s is close to an optimum. As for the value of λ, it turns out that $\lambda = 0.85$ is probably optimal. (In the 1977-paper of one of the present authors, we put $\lambda = 0.2$ following Cederwall and Ditmars (1970), but this is far too low.) There is a general tendency in the energy-approach that the centerline velocities are predicted to be high. The deviations, noted in the 1977-paper, still persist when λ is replaced by the correct value of 0.85. However, it should be noted here that the polynomial solution of Fanneløp et al. (1991) leads to even higher theoretical centerline velocities. Perhaps these deviations have to do with the fact that a plume fed from a line source is after all not strictly two-dimensional. Although the measurements are somewhat uncertain we may conclude that the energy approach, predicting actually lower centerline velocities than the entrainment approach, should be seriously considered as a viable alternative.

3. The flow far outside a plane plume

We now leave the entrainment coefficient α, and consider the field – mostly the horizontal fluid velocity – far away from a plane plume. By "far" field we mean positions away from the turning region; in practice, this means horizontal distances $x > 0.5D$. Experimental evidence shows that the horizontal range of influence from a plane plume extends out to about 2.5–7 depths. This large variation was pointed out by Riess and Fanneløp (1998) and was found to be partly due to different experimental geometries and different definitions of the range of influence. We will below present a model for the far field that is essentially an extension of the model given earlier by Riess and Fanneløp.

3.1. The model of Riess and Fanneløp

The thesis of Riess (1997), and the related paper of Riess and Fanneløp (1998), consider a theory for plane plumes based upon plane turbulent jet theory. The horizontal velocity profile is

$$u(z,x) = \frac{m}{\sqrt{x}} \exp\left(-\frac{z^2}{k^2x^2}\right), \tag{13}$$

where m and k are momentum and spreading coefficients. Here and henceforth, the z axis points downward from the free surface. A return flow, assumed homogeneous with respect to z, is found by integrating the above profile from the sea floor to the surface. This return flow follows from requiring continuity of the flow. Non-dimensional versions of the velocity components u and w are called u^* and w^*, where u^* is given by Eq. (5) and

$$w^*(z,x) = \frac{1}{4}km^*\left(\frac{D}{x}\right)^{3/2}\left[-\frac{x\sqrt{\pi}}{D}\operatorname{erf}\left(\frac{z}{kx}\right) + \frac{zx\sqrt{\pi}}{D^2}\operatorname{erf}\left(\frac{D}{kx}\right) + \frac{4z}{Dk}\left(\exp\left(-\frac{z^2}{k^2x^2}\right) - \exp\left(-\frac{D^2}{k^2x^2}\right)\right)\right], \tag{14}$$

with $m^* = m/(w_p\sqrt{D})$. Further, $u = w_p u^*$ and $w = w_p w^*$ where, as mentioned in Section 2.1, w_p is a scaling parameter equal to the centerline velocity at the surface.

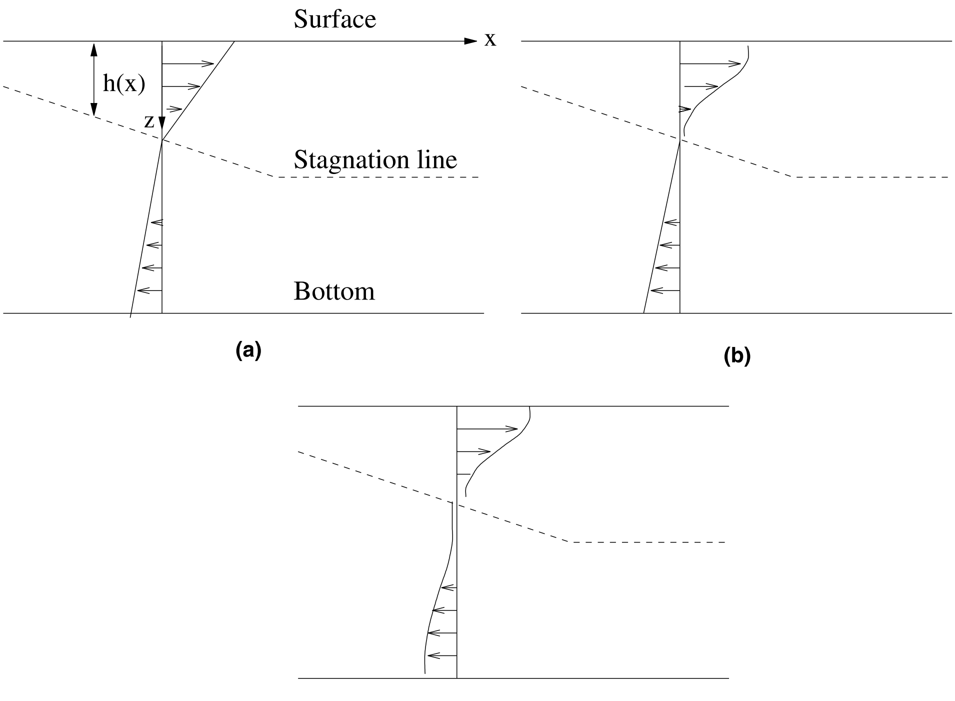

Fig. 2. Three options for the velocity profiles, above and below the stagnation line. Alternative (c) is chosen.

This simple model is said to be valid for shallow depths. It has, however, the drawback that the coefficients m^* and k must be found from least square fits to measurements. The coefficients were determined only for one case, viz. $D = 1$ m, $Q^0 = 3.41 \times 10^{-3}$ m^2/s, leading to $m^* = 1.0$, $k = 0.18$.

A plot of this case (not shown here) shows that the theoretical predictions resemble the observed data, although the surface velocities near the plume are predicted somewhat high. For example, at a position 5 cm below the free surface and 0.5 m away from the plume center the model predicts a velocity of 0.62 m/s, about 30% in excess of the measured value (0.47 m/s measured from figure in Fanneløp et al.). The most likely explanation for this discrepancy is that the parameter w_p is overestimated because the Riess–Fanneløp model neglects the slip velocity. In addition, one must of course be aware of experimental uncertainties when assessing discrepancies between theory and experiment. However, in spite of unknown experimental errors, as well as the obvious fact that the bubble plume does not satisfy strict two-dimensionality, it becomes very natural to try to improve on one specific point in the Riess–Fanneløp theory, namely its assumption about a *uniform* return flow. A uniform flow model is very crude; it implies subtracting off a reverse current which is uniform up to the free surface. This is the topic of the following section.

3.2. A non-uniform return flow model

As above, we begin by letting the plane flow nearest to the line source be represented by an outflowing jet. Different profiles for the non-uniform return flow were tested and are shown in Fig. 2. We chose the third alternative with Gaussian profiles above and below the stagnation line, since this should be most physical. However, the upper part of the return flow looks different in experiments (see Wen and Torrest, 1987). Based upon Fig. 8(a) in Fanneløp et al. (1991), we propose the following form for the stagnation line:

$$h(x)/D = \begin{cases} 0.125(1 + x/D), & x \leqslant 3D, \\ 0.5, & x > 3D, \end{cases} \tag{15}$$

where $h(x)$ is the depth of the outflowing layer. This form fits well with Fanneløp et al. ($D = 1$ m), and also with Bulson's measurements ($D = 7.8$ and 8.9 m) at horizontal distance $x = D$ (only 7% error). We assume tentatively that Eq. (15) holds for other depths also. However, Wen and Torrest (1987) published a different formula for the stagnation line

$$\frac{h(x)}{D} = 0.05\frac{x}{D} + 0.308$$

using manifold depths between 25 and 62.5 cm. Both the data in Fanneløp et al. (1991) and Wen and Torrest (1987) imply that the stagnation line is almost independent of air flow rate, but taken together, they imply that there may be a dependence on D rather than just x/D.

An initial condition has to be found to estimate the strength of the surface jet. From turbulent jet theory one knows that the surface velocity is $u_0 \equiv u(0, x) = m/\sqrt{x}$ (it corresponds to setting $z = 0$ in Eq. (13)). The constant m is unknown, but can be found by using Eq. (35) in Brevik (1977), which says that the maximum horizontal surface velocity in a plane plume is

$u_{\max} = 1.7(gQ^0)^{1/3}(1 + D/P)^{-1/3}$. Experimentally it turns out that the maximum surface velocity occurs at a distance $x = 0.6D$ from the plume. We can accordingly determine m from the relation

$$\frac{m}{\sqrt{0.6D}} = 1.7(gQ^0)^{1/3}\left(\frac{P}{D+P}\right)^{1/3}. \tag{16}$$

Next, again using Fig. 8(a) in Fanneløp et al. (1991), we see that the layer depth h corresponds to two standard deviations for the approximate Gaussian profile for the surface flow. Similarly, $(D - h)$ is seen to correspond to about two standard deviations in the return flow profile. Altogether, if we let U_1 and U_2 denote the horizontal velocity components respectively above and below the stagnation line, we get in the region $x \leqslant 3D$

$$U_1 = u_0 \exp\left(-\frac{2z^2}{h^2}\right), \tag{17}$$

$$U_2 = -\frac{u_0 h}{D-h} \exp\left[-2\left(\frac{1-z/D}{1-h/D}\right)^2\right]. \tag{18}$$

The prefactor in Eq. (18) follows from the requirement that no net flow goes through a vertical cross section parallel to the plume plane.

For $x > 3D$ we find in the same way

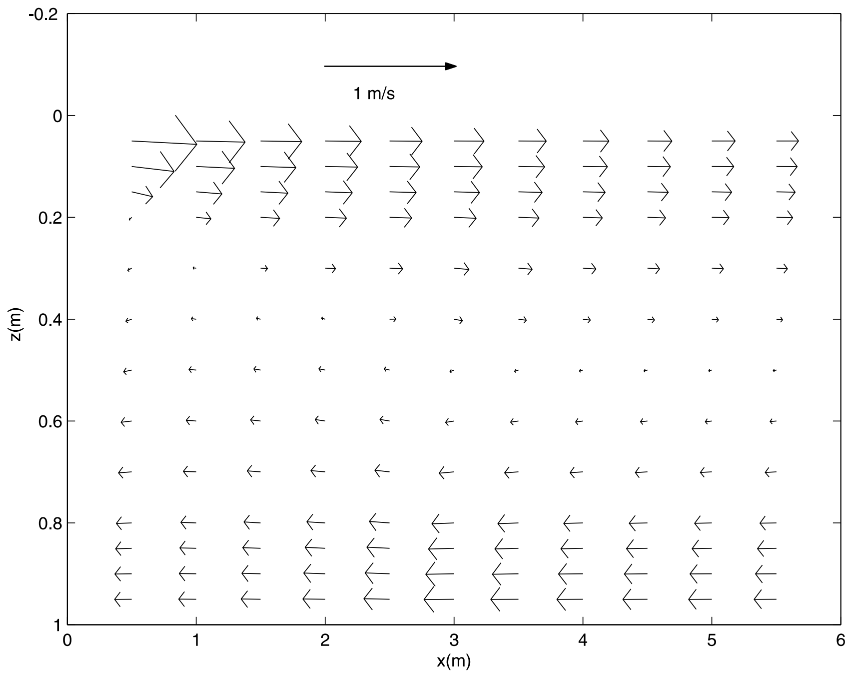

Fig. 3. Plot of the flow field as calculated from the model of Section 3.2. $D = 1$ m, $Q^0 = 3.41 \times 10^{-3}$ m^2/s.

$$U_1 = u_0 \exp\left(-\frac{8z^2}{D^2}\right), \tag{19}$$

$$U_2 = -u_0 \exp\left[-8\left(1-\frac{z}{D}\right)^2\right]. \tag{20}$$

From the equation of continuity, the vertical velocity components W_1 and W_2 above and below the stagnation line can now be calculated. These expressions turn out to be more complicated than the expressions for U_1 and U_2, and are also of less physical significance. They are therefore omitted here. The predicted currents are shown in Fig. 3. (This figure also incorporates the contributions from calculated values of W_1 and W_2.) Note that we need now not any information about m^* and k; what we need instead, is the jet strength m which is easily calculable from Eq. (16).

The velocity at 5 cm below the free surface and 0.5 m away from the plume which, as mentioned earlier, was measured to be 0.47 m/s, is predicted to be 0.50 m/s by this new model. It is thus closer to experiment than the value 0.62 m/s following from the theory of Riess and Fanneløp. The improvements in the theory are most pronounced close to the top of the plume.

As mentioned above, the horizontal range of influence is about 2.5–7D, according to experiments. This is seen to be in reasonable agreement with the predictions of Fig. 3, but our model somewhat overestimates the range of influence. Another point worth noticing is that the horizontal velocity profile near to the plume is almost *linear*. This important property, noted earlier by Taylor (1955) in his study of hydraulic and pneumatic breakwaters, was made use of also in similar studies by Brevik (1976a,b).

4. The surface flow close to the plume: an energy approach

We now leave the far field, and consider instead the surface flow close to the plume.

Our motivation for undertaking this study is the following. Earlier studies of the surface flow, of Fanneløp and Sjøen (1980) in the axisymmetric case and of Fanneløp et al. (1991) in the plane case, were based on the assumption that the fluid *momentum* be conserved in the turning region. This assumption is however not correct, for fundamental reasons: there are separate equations for conservation of momentum in the vertical, and in the horizontal direction, but one cannot from these equations draw any conclusion about conservation of momentum when the plume turns from the vertical to the horizontal. It lies at hand therefore to try replacing the momentum conservation assumption by another assumption being physically more plausible. We will below replace the momentum equation by an equation expressing conservation of *kinetic energy*, account being taken of turbulent dissipation through a dissipation factor f. In this way we do not break any fundamental conservation laws. (Our energy flux method is rather rough, though, in the sense that the influence from the pressure term in the energy flux is not taken into account explicitly. Its influence is assumed to be dealt with implicitly, through the f factor.)

4.1. Axisymmetric plumes

In Fanneløp and Sjøen (1980) the surface flow was assumed to have a Gaussian profile

$$u(z,r) = u_0(r)\exp\left[-\frac{z^2}{h_w^2(r)}\right]. \tag{21}$$

where again the z axis points downward from the free surface. This corresponds to the horizontal mass flux

$$\dot{m}(r) = \rho_w 2\pi r \int_0^\infty u(z,r)\,dz = \pi^{3/2}\rho_w h_w u_0 r. \tag{22}$$

According to the entrainment hypothesis the rate of entrainment is proportional to the surface velocity, the contact area $2\pi r$ and the entrainment coefficient β:

$$\frac{d\dot{m}(r)}{dr} = 2\pi r\beta\rho_w u_0. \tag{23}$$

We choose to apply the initial conditions at the position $r = 2b_p$, where b_p is the $1/e$-width of the vertical plume velocity profile at top of the plume. The vertical mass flux at this position is

$$\dot{m}_p = \rho_p \int_0^\infty w_p \exp\left(-\frac{r^2}{b_p^2}\right) 2\pi r\,dr = \pi\rho_p w_p b_p^2, \tag{24}$$

where w_p is the centerline velocity of the plume at the surface. Using continuity of mass flux and the Boussinesq approximation $\rho_p \approx \rho_w$ we get the following initial condition for the surface flow:

$$\left(\pi^{3/2}\rho_w h_w u_0 r\right)\Big|_{r=2b_p} = \pi\rho_w w_p b_p^2. \tag{25}$$

We analyse the energy flux in a similar way: the kinetic energy flux $\dot{K}_w$ in the surface flow is

$$\dot{K}_w(r) = \frac{1}{2}\rho_w \cdot 2\pi r \int_0^\infty u^3(z,r)\,dz = \frac{\pi^{3/2}}{2\sqrt{3}}\rho_w h_w u_0^3 r, \tag{26}$$

which is to be related to the kinetic energy flux at top of the plume

$$\dot{K}_p = \frac{1}{2}\rho_p \int_0^\infty w_p^3 \exp\left(-\frac{3r^2}{b_p^2}\right) 2\pi r\,dr = \frac{\pi}{6}\rho_p w_p^3 b_p^2 \tag{27}$$

through the proportionality factor f, i.e., $\dot{K}_w = f\dot{K}_p$. Again putting $\rho_p \approx \rho_w$ and combining this equation with Eq. (24), we obtain the following differential equation for $u_0(r)$:

$$\frac{du_0}{dr} + \frac{\sqrt{3}\beta}{w_p^3 b_p^2 f} r u_0^4 = 0. \tag{28}$$

Integrating this equation, determining the integration constant via Eq. (25), we obtain

$$u_0(r) = w_p\left(\frac{f}{\sqrt{3}\beta}\right)^{1/3}\left(\frac{3}{2b_p^2}r^2 + \frac{3^{1/4}f^{-1/2} - 6\beta}{\beta}\right)^{-1/3}, \tag{29}$$

$$h_w(r) = \frac{b_p^2\beta}{\sqrt{\pi}}\frac{1}{r}\left(\frac{3}{2b_p^2}r^2 + \frac{3^{1/4}f^{-1/2} - 6\beta}{\beta}\right). \tag{30}$$

Here w_p and b_p are to be determined from the plume solution, whereas values for the entrainment factor β and the dissipation factor f must be estimated.

We have calculated the predictions of this new theory for a number of cases and compared with axisymmetric experiments. The various experimental results make us conclude, as an average, that the appropriate value of β is about 0.08, perhaps as large as 0.10 in some cases. Moreover, the typical value of the turbulent dissipation factor f is about 0.7, perhaps as large as 0.8.

As an example, Fig. 4 shows a comparison of our theory with Goossens' measurements at a source depth of 4 m in a cylindrical tank, when $Q^0 = 9.3 \times 10^{-4}$ m^3/s. In this case, the values $\beta = 0.08, f = 0.8$ were found to be good, the sensitivity with respect to f being low. It is here to be noted that there is still one-third empirical coefficient to be accounted for, namely the *slip correction*. According to simulations by Milgram (1983), centerline velocities are reduced by about 17% when the slip velocity increases from zero to realistic values, whereas according to Fanneløp and Sjøen (1980) the correction is about 15%. Besides, the centerline velocity w_p is evaluated not at the top of the plume but about $0.07D$ below the surface. Altogether, we have chosen the slip correction to be 20% in Fig. 4, suggesting this value to be rather universal. The agreement between theory and experiment is seen to be good.

We have made similar comparisons also with the large scale experiment of Goossens, $D = 15$ m. The agreement was found to be comparable, though the scatter became greater, as expected under natural conditions.

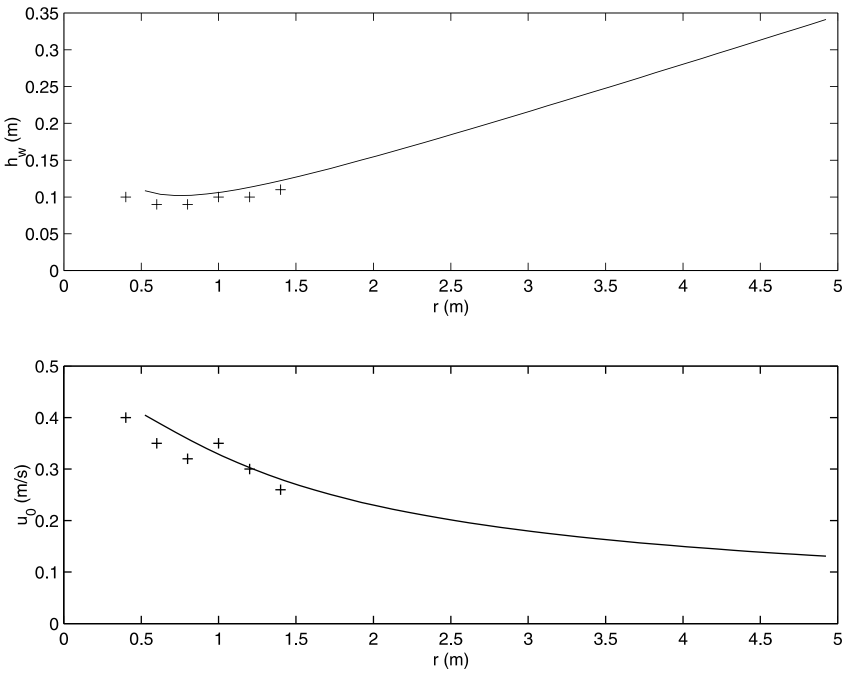

Fig. 4. Axisymmetric plume: plot of the flow field as calculated from Eqs. (29) and (30), versus Goossens' tank measurements. $D = 4$ m, $Q^0 = 4.3 \times 10^{-4}$ m^3/s. Input parameters are $\beta = 0.08$, $f = 0.8$, slip correction 20%.

4.2. Plane plumes

The plane plume can be treated in a similar way. The theory of Fanneløp et al. (1991) is modified, in the sense that the continuity condition on the momentum flux in the turning region is replaced by a continuity condition on the energy flux.

A sketch of the situation is given in Fig. 5. The present case differs from the axisymmetric model of the previous section in that a linear profile is used for the outward moving jet. This choice is made to facilitate the comparison between theoretical and empirical values of $h_\mathrm{w}(x)$. The vertical plume, as before, is assumed to have a Gaussian profile, and the initial condition, following Fanneløp et al. (1991), is applied at position $x = b_\mathrm{p}$, where b_p is the Gaussian plume width at the surface. The jet profile is

$$u(z,x) = u_0(x)\left(1 - \frac{z}{h_\mathrm{w}(x)}\right). \tag{31}$$

It corresponds to the horizontal mass flux $\dot{m}(x) = (\rho_\mathrm{w}/2)h_\mathrm{w}u_0$ per unit length of the pipe. According to the entrainment hypothesis

$$\frac{\mathrm{d}\dot{m}(x)}{\mathrm{d}x} = \beta\rho_\mathrm{w}u_0. \tag{32}$$

Now proceeding in a manner similar to the previous section (details omitted here), we derive the differential equation

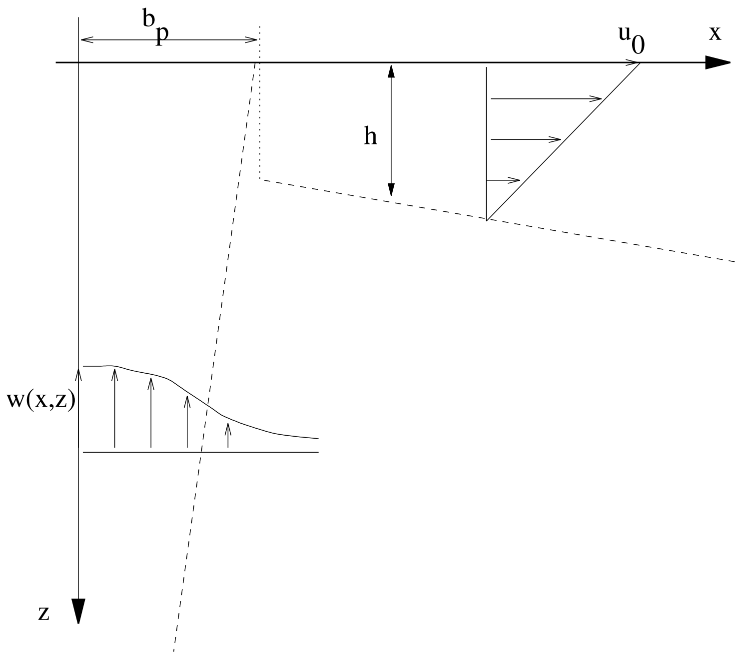

Fig. 5. Sketch of the plane-plume flow near the turning region.

$$\frac{\mathrm{d}u_0}{\mathrm{d}x} + \frac{\beta}{a} u_0^4 = 0 \quad \text{with} \ a = f\sqrt{\frac{4\pi}{3}} w_\mathrm{p}^3 b_\mathrm{p}, \tag{33}$$

which leads to the solutions

$$u_0(x) = w_\mathrm{p}\left(\frac{2f}{\sqrt{3}}\right)^{1/3}\left[\frac{3\beta}{\sqrt{\pi}}\left(\frac{x}{b_\mathrm{p}} - 1\right) + \frac{3^{1/4}}{\sqrt{2f}}\right]^{-1/3}, \tag{34}$$

$$h_\mathrm{w}(x) = 3\beta\left(x - b_\mathrm{p}\right) + b_\mathrm{p}3^{1/4}\sqrt{\frac{\pi}{2f}}. \tag{35}$$

Results calculated from these equations are shown in Fig. 6. The theoretical curves are based on the plume solutions of Brevik (1977), using $\lambda = 0.85$, and on the polynomial solutions of Fanneløp et al. (1991). Also, the empirical stagnation line, as given by Eq. (15), is plotted. The parameter β was determined from the slope of h_w close to the plume, and gave the value $\beta = 0.06$. The optimum value of the dissipation factor f was about 0.8. Both these values are physically reasonable.

It is apparent that we get good results in the region $x < 2D$, especially when using the plume solution of Fanneløp et al. (without slip correction). We also made comparisons between the present theory and Bulson's measurements. Even in this deep-water case, implying a gas flow rate 20 times higher and a depth up to five times higher than the case of Fig. 6, we found good

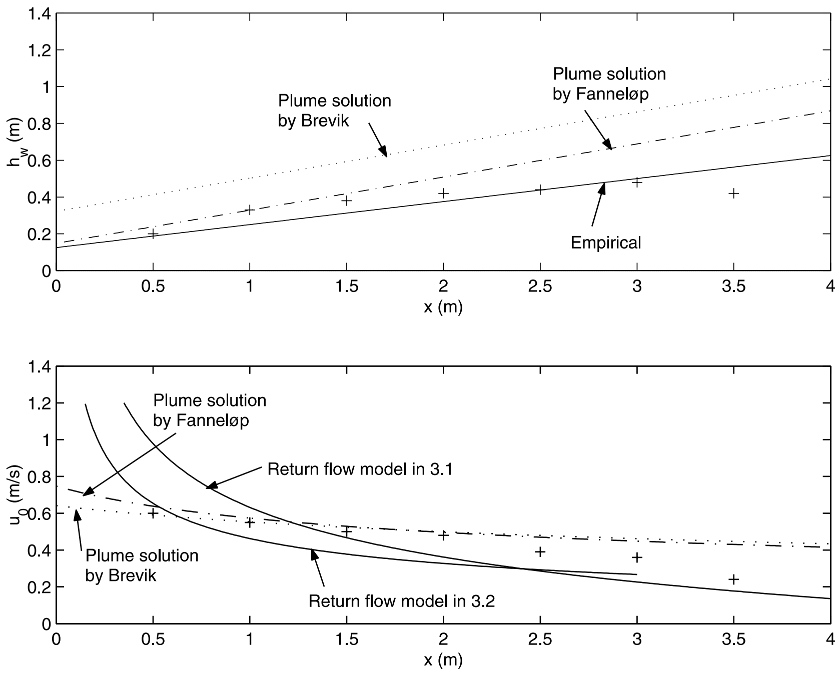

Fig. 6. Plane plume: depth h_w of surface flow and surface velocity u_0 based upon the models of Brevik (1977) and Fanneløp et al. (1991). "Empirical" means the stagnation line. $D = 1$ m, $Q^0 = 4.87 \times 10^{-3}$ m$^2/3$, $\beta = 0.06$, $f = 0.8$.

agreement for $x < 2D$. The same values of β and f were applicable, thus supporting the general usefulness of the present model.

5. Final remarks

A general summary of our work is given at the end of Section 1. The air-bubble system is obviously a very complex phenomenon. Needless to say, we have not been able to deal with all aspects of it in our treatment above. Let us yet close our paper by briefly discussing a few points of physical interest:

(1) First, one should distinguish between the dominant phenomena in laboratory-scale experiments and in full scale. At the smallest depths, compressibility is rather unimportant. The rise velocity of single bubbles in quiescent water is of the same order of magnitude as the bulk water velocity and slip cannot be neglected. There are two sources of turbulence, viz. the interacting bubble wakes and the shear associated with the plume flow. The length scales are here different, but much closer in size than in full scale tests (tens of meters). In full scale, compressibility is of prime concern and the increased buoyancy leads to accelerated flow in certain regimes. Bubble slip is of minor importance, as the bubbles remain small on average. The most important turbulent quantities are most likely those associated with the shear flow of scale of the same order as the plume radius. The conclusion to be drawn is: care must be taken where a model validated by laboratory experiments is to be applied under natural conditions.

(2) Our second remark relates to the turning region, where the water flow in the plume turns by 90° over a length of the order of the plume radius. This region is rather complex: the surface is lifted up to a height corresponding to the stagnation pressure. When bubbles break through this surface there is considerable splashing with small fountains reaching heights many times that of the average potential height. This leads to a loss of energy that may in turn affect the strength of the outward directed flow along the surface (this effect is in our approach dealt with through the dissipation factor f, in Section 4). Such phenomena have been dealt with; cf., for instance, Engebretsen et al. (1997) and Friedl (1998). Engebretsen et al. noted that in their experiments with large discharges, the flow from the fountain penetrated the deflected surface flow and contributed to a deeper surface velocity profile. This effect is not dealt with in Section 4, but one should keep in mind that their experiments were made in a tank with a small surface compared to the depth (6×9 m^2 and 7 m deep). Also, in the experiments with only underwater measurements, the duration of the release was limited to 20 s.

(3) We assumed above that the water surrounding the plume is homogeneous. A special variant of the plume problem occurs when the surroundings are stratified. Such plumes have been studied by Imberger and co-workers, cf., for instance, Asaeda and Imberger (1993).

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